Solution by Method of Variation of Parameter

Let us consider the non homogeneous second order Linear DE with constant co-efficient

$$y'' + ay' + by = f(x) \quad \text{——— (1)}$$

or $$x^2y'' + axy' + by = f(x) \quad \text{——— (2)}$$

STEPS FOR FINDING OUT THE GENERAL SOLUTION BY USING METHOD OF VARIATION OF PARAMETER

STEP:-I

Find out the complementary function y_c of equation (1) or (2) by writing the homogeneous form of (1) or (2) which one is given.

It is always in the form of $y_c = c_1y_1 + c_2y_2$.

STEP:-II

consider the particular integral y_p as

$$y_p = c_1(x)\, y_1 + c_2(x)\, y_2 \quad \text{——— (3)}$$

by converting the constants c_1 & c_2 as functions of x i.e $c_1(x)$ & $c_2(x)$

STEP:-III

Find out the Wronskian W by using the formula

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

STEP:-IV

Calculate the value of $c_1(x)$ and $c_2(x)$ by using the formula

$$c_1(x) = -\int \frac{y_2 f(x)}{W} dx$$

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$$C_2(x) = \int \frac{y_1 f(x)}{W} dx .$$

STEP:-V

Put the value of $C_1(x)$ and $C_2(x)$ in equation (3) to get the value of y_p.

STEP:-VI

The general solution $y = y_c + y_p$.

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NO-1-TO-10

Find a general solution by using the method of variation of parameter.

1. $y'' - 4y' + 4y = \frac{e^{2x}}{x}$ ——— (1)

Ans:- The homogeneous form of equation (1) is

$$y'' - 4y' + 4y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$$\Rightarrow (D^2 - 4D + 4)y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

The auxilary equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2, 2$$

So, $y_c = (C_1 + C_2 x) e^{2x}$

Let us consider $y_p = C_1(x) e^{2x} + C_2(x) x e^{2x}$

Here, $y_1 = e^{2x}$ $\quad y_2 = x e^{2x}$

$y_1' = 2e^{2x}$ $\quad y_2' = e^{2x} + 2x e^{2x}$

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The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

$$C_1(x) = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{x e^{2x} \frac{e^{2x}}{x}}{e^{4x}} dx$$

$$= -\int dx = -x$$

$$C_2(x) = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^{2x} \frac{e^{2x}}{x}}{e^{4x}} dx = \int \frac{dx}{x}$$

$$= \ln|x|$$

So, $y_p = -x e^{2x} + (\ln|x|) x e^{2x}$

Hence, the general solution

$$y = y_c + y_p$$

$$\Rightarrow y = (C_1 + C_2 x) e^{2x} - xe^{2x} + x e^{2x} \ln|x|$$

2. $y'' + 9y = \sec 3x$ ——(1)

Ans:- The homogeneous form of equation (1) is

$$y'' + 9y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 9y = 0$$

$$\Rightarrow (D^2 + 9) y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

The Auxilary equation is

$$\lambda^2 + 9 = 0$$

$$\Rightarrow \lambda^2 = -9 = 9i^2$$

$$\Rightarrow \lambda = \pm 3i$$

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So, $y_c = e^{0x}(C_1 \cos 3x + C_2 \sin 3x)$

$\Rightarrow y_c = C_1 \cos 3x + C_2 \sin 3x$.

Let $y_p = C_1(x)\cos 3x + C_2(x)\sin 3x$

Here, $y_1 = \cos 3x \qquad y_2 = \sin 3x$

$y_1' = -3\sin 3x \qquad y_2' = 3\cos 3x$

The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$

$= 3\cos^2 3x + 3\sin^2 3x$

$= 3$.

$$C_1(x) = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin 3x \sec 3x}{3} dx$$

$$= -\frac{1}{3}\int \tan 3x \, dx$$

$$= -\frac{1}{3}\cdot\frac{1}{3}\ln|\sec 3x|$$

$$= -\frac{1}{9}\ln|\sec 3x|$$

$$C_2(x) = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos 3x \sec 3x}{3} dx$$

$$= \int \frac{dx}{3} = \frac{x}{3} .$$

So, $y_p = -\frac{1}{9}\ln|\sec 3x|\cos 3x + \frac{x}{3}\sin 3x$

Hence, the general solution

$y = y_c + y_p$

$\Rightarrow y = C_1\cos 3x + C_2\sin 3x - \frac{1}{9}\ln|\sec 3x|\cos 3x + \frac{x}{3}\sin 3x$.

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11-TO-17

Find a general solution by using the method of variation of parameter.

11. $x^2y'' - 4xy' + 6y = 21x^{-4}$ ——— (1)

Ans:-

The homogeneous form of equation (1) is

$$x^2y'' - 4xy' + 6y = 0$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0$$

$$\Rightarrow (x^2D^2 - 4xD + 6)y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

——— (2)

Put $y = x^m$

$$y' = Dy = mx^{m-1}$$

$$y'' = D^2y = m(m-1)x^{m-2}$$

Putting these values in equation (2)

$$[m(m-1) - 4m + 6]x^m = 0$$

The auxilary equation is

$$m(m-1) - 4m + 6 = 0$$

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

So, $y_c = C_1x^2 + C_2x^3$

Let $y_p = C_1(x)x^2 + C_2(x)x^3$

Here, $y_1 = x^2$ $\qquad y_2 = x^3$

$y_1' = 2x$ $\qquad y_2' = 3x^2$

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The Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

$$C_1(x) = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{x^3 \cdot \frac{21x^{-4}}{x^2}}{x^4} dx$$

$$= -21 \int x^{-7} dx = -21 \frac{x^{-6}}{-6}$$

$$= \frac{7}{2} x^{-6}$$

$$C_2(x) = \int \frac{y_1 f(x)}{W} dx = \int \frac{x^2 \cdot \frac{21x^{-4}}{x^2}}{x^4} dx$$

$$= 21 \int x^{-8} dx = 21 \frac{x^{-7}}{-7} = -3x^{-7}$$

Now $y_p = \frac{7}{2} x^{-6} x^2 - 3x^{-7} x^3$

$= \frac{7}{2} x^{-4} - 3x^{-4} = \frac{1}{2} x^{-4}$

The general solution

$y = y_c + y_p$

$\Rightarrow y = C_1 x^2 + C_2 x^3 + \frac{1}{2} x^{-4}$.

17. $(x^2D^2 + xD - 9) y = 48x^5$ ——— (1)

Ans:- The homogeneous form of equation (1) is

$(x^2D^2 + xD - 9) y = 0$ ——— (2)

Put $y = x^m$

$y' = Dy = mx^{m-1}$

$y'' = D^2y = m(m-1)x^{m-2}$

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Putting these values in equation (2)

$$m(m-1) + m - 9 = 0$$

$$\Rightarrow m^2 - 9 = 0$$

$$\Rightarrow (m+3)(m-3) = 0$$

$$\Rightarrow m = -3, 3$$

So, $y_c = c_1 x^{-3} + c_2 x^3$

Let $y_p = c_1(x)\, x^{-3} + c_2(x)\, x^3$

Here, $y_1 = x^{-3}$ $\quad y_2 = x^3$

$y_1' = -3x^{-4}$ $\quad y_2' = 3x^2$

The Wronskeian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-3} & x^3 \\ -3x^{-4} & 3x^2 \end{vmatrix}$$

$$= 3x^{-1} + 3x^{-1} = 6x^{-1}$$

$$c_1(x) = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{x^3 \, \frac{48x^5}{x^2}}{6x^{-1}} dx$$

$$= -8\int x^7 dx = -8\,\frac{x^8}{8} = -x^8$$

$$c_2(x) = \int \frac{y_1 f(x)}{W} dx = \int \frac{x^{-3} \, \frac{48x^5}{x^2}}{6x^{-1}} dx = 8\int x\, dx = 4x^2$$

So, $y_p = -x^8 x^{-3} + 4x^2 x^3$

$$= -x^5 + 4x^5$$

$$= 3x^5.$$

Hence the general solution

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 x^{-3} + c_2 x^3 + 3x^5.$$

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ASSIGNMENT-1

1. Find a general solution

(a) $xy' = y^2 + y$

(b) $y' = \dfrac{1 - 2y - 4x}{1 + y + 2x}$

2. Solve

(a) $xy\,dy = (x^2 + y^2)\,dx$

(b) $xy\,y' = 2y^2 + 4x^2$, $y(2) = 4$

3. Find a general solution

(a) $(\cot y + x^2)\,dx = x \operatorname{cosec}^2 y\,dy$

(b) $y e^x dx + (2y + e^x)\,dy = 0$

4. Find an integrating factor and solve

(a) $(2\cos y + 4x^2)\,dx = x \sin y\,dy$

(b) $(y + 1)\,dx - (x + 1)\,dy = 0$

5. Solve the initial value problem

(a) $y' + 6x^2 y = \dfrac{e^{-2x^3}}{x^2}$, $y(1) = 0$

(b) $y' - (1 + 3x^{-1})y = x + 2$ $y(1) = e - 1$

6. Solve

(a) $y' + y = -\dfrac{x}{y}$

(b) $y' + 2y = y^2$

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